Matlab Programs for Computation of Isentropic Compressible Flow

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February 23, 2000

This document briefly summarizes a set of Matlab programs for computing flow properties for the one-dimensional isentropic flow of an ideal gas. The nomenclature and sign conventions used here are consistent with the textbooks by Munson, Young and Okiishi [1], and White [2].

Governing Equations

Stagnation Properties as Functions of Ma

For one-dimensional, compressible, isentropic flow of and ideal gas the following equations relate the static properties, p, T, and ρ to the *stagnation* properties, p_0, T_0, and ρ_0.

$$\frac{p}{p_0} = \left[\frac{1}{1 + \dfrac{k-1}{2} Ma^2}\right]^{k/(k-1)} \tag{1}$$

$$\frac{T}{T_0} = \frac{1}{1 + \dfrac{k-1}{2} Ma^2} \tag{2}$$

$$\frac{\rho}{\rho_0} = \left[\frac{1}{1 + \dfrac{k-1}{2} Ma^2}\right]^{1/(k-1)} \tag{3}$$

where $k = c_p/c_v$ is the specific heat ratio and $Ma = V/\sqrt{kRT}$ is the Mach number.

Duct Area Relationship for a Converging-Diverging Nozzle

For isentropic flow in a converging-diverging nozzle the ratio of local duct area to the area at the throat is uniquely related to the value of Ma. If A^* is the area of the duct section where $Ma = 1$, then the area at any other section along a converging-diverging nozzle is related to Ma by

$$\frac{A}{A^*} = \frac{1}{Ma}\left[\frac{1 + \dfrac{k-1}{2} Ma^2}{1 + \dfrac{k-1}{2}}\right]^{(k+1)/[2(k-1)]} \tag{4}$$

Note that this ratio may be computed even if the flow is not sonic at the minimum *physical* area. In that case A^* is a reference value of the area, not the actual area of the duct at a particular section.

Ma as a Function of Stagnation Properties

If Ma is unknown, but one of the preceding stagnation property ratios is known, then Ma may be computed. Solving equation (1) through (3) for Ma gives

$$Ma = \sqrt{\frac{2}{k-1}\left[\left(\frac{p}{p_0}\right)^{(1-k)/k} - 1\right]} \tag{5}$$

$$Ma = \sqrt{\frac{2}{k-1}\left[\frac{T_0}{T} - 1\right]} \tag{6}$$

$$Ma = \sqrt{\frac{2}{k-1}\left[\left(\frac{\rho}{\rho_0}\right)^{1-k} - 1\right]} \tag{7}$$

Ma as a Function of Area Ratio

Equation (4) cannot be solved for Ma. If A/A^* is known equation (4) can be used in a root-finding procedure to obtain a numerical value of Ma that satisfies the equation. Rewriting equation (4) as

$$f\left(\frac{A}{A^*}\right) = \frac{A}{A^*} - \frac{1}{Ma}\left[\frac{1+\dfrac{k-1}{2}Ma^2}{1+\dfrac{k-1}{2}}\right]^{(k+1)/[2(k-1)]} \tag{8}$$

gives an equation suitable for use with the built-in `fzero` function. When the correct value of A/A^* is guessed (for given values of Ma and k) then $f(A/A^*) = 0$.

M-files

Table 1 lists the MATLAB functions that implement the computations outlined in the preceding equations.

Examples

Compute the property ratios T/T_0, p/p_0, and ρ/ρ_0 for air at $Ma = 0.75$

```
>> isenTT0(0.75)
ans =
    0.8989

>> isenpp0(0.75)
ans =
    0.6886

>> isenrr0(0.75)
ans =
    0.7660
```

Repeat the preceding calculations at $Ma = 0.75$ for Helium ($k = 1.66$) instead of air

```
>> isenTT0(0.75,1.66)
ans =
    0.8434

>> isenpp0(0.7,1.665)
ans =
    0.6516

>> isenrr0(0.75,1.66)
ans =
    0.7726
```

Now, assume that the stagnation properties are known, but Ma is not. If $T/T_0 = 0.5$ for air, the value of Ma is

```
>> isenMaTT0(0.5)
ans =
    2.2361

>> isenTT0(ans)                  % check preceding calculation
ans =
    0.5000
```

The call to `isenTT0` reverses the computation of Ma, thereby providing a check on the calculations in the m-file. (See `testIsenProps` for a complete set of tests.)

For $Ma = 0.75$ and $Ma = 1.5$ the area ratio in equation (4) is computed with

```
>> isenAAs(0.75)
ans =
    1.0624

>> isenAAs(1.5)
ans =
    1.1762
```

The inverse computation is handled by the `isenMaaas` function.

```
>> isenMaaas(1.0624)
ans =
    0.7500

>> isenMaaas(1.1762)
ans =
    0.6104
```

This last result appears to be in error, but it is not. By default, `isenMaaas` returns the subsonic Ma that satisfies equation (4) for a given value of A/A^*. For a given A/A^* both subsonic ($Ma < 1$) and supersonic ($Ma > 1$) solutions are possible. To select the supersonic solution a second input to the `isenMaaas` function is needed. Only the sign of the second argument is important: if the second argument is negative the subsonic branch is chosen, if it is positive the supersonic branch is chosen. Thus

```
>> isenMaaas(1.1762,1)
ans =
    1.5000
```

confirms that `isenAAs` and `isenMaaas` are working correctly.

The `testIsenProps`, `MYO_11_38`, and `White_E9_3` functions provide additional examples of using the m-file functions in this toolbox.

References

1. B.R. Munson, D.F. Young, and T.H. Okiishi, *Fundamentals of Fluid Mechanics*, third edition, 1998, Wiley, New York

2. F.M. White, *Fluid Mechanics*, fourth edition, 1999, McGraw-Hill, New York

Function	equation	Description
`aasmaResidual`	(8)	Evaluates equation (8) for use with a root-finding procedure for finding Ma as a function of A/A^*.
`MYO_11_38`	N.A.	Computations used in solution to problem 11.38 in Munson, Young and Okiishi.
`isenAAs`	(4)	Area ratio A/A^* for isentropic compressible flow
`isenMaaas`	N.A.	Ma as a function of area ratio A/A^* for isentropic compressible flow. Computing Ma requires a root-finding procedure so the equation for A/A^* as a function of Ma cannot be written explicitly. `isenMaas` uses the built-in `fzero` function and the `aasmaResidual` function to find the value of A/A^* that gives $f(A/A^*) = 0$ in equation (8).
`isenMapp0`	(5)	Ma as a function of pressure ratio p/p_0 for isentropic compressible flow
`isenMarr0`	(7)	Ma as a function of density ratio ρ/ρ_0 for isentropic compressible flow
`isenMaTT0`	(6)	Ma as a function of temperature ratio T/T_0 for isentropic compressible flow
`isenpp0`	(1)	Pressure ratio p/p_0 for isentropic compressible flow
`isenrr0`	(3)	Density ratio ρ/ρ_0 for isentropic compressible flow
`isenTT0`	(2)	Temperature ratio T/T_0 for isentropic compressible flow
`testIsenProps`	N.A.	Test all routines in this toolbox
`White_E9_3`	N.A.	Computations used in Example 9.3 in White.

Table 1: Functions for computing isentropic flow properties for one-dimensional compressible flow of an ideal gas.