Numerical Integration of Ordinary Differential Equations for Initial Value Problems

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The latest version of this PDF file, along with other supplemental material for the book, can be found at www.prenhall.com/recktenwald or web.cecs.pdx.edu/~gerry/nmm/.

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Overview

- Motivation: ODE's arise as models of many applications
- Euler's method
  - ▷ A low accuracy prototype for other methods
  - ▷ Development
  - ▷ Implementation
  - ▷ Analysis
- Midpoint method
- Heun's method
- Runge-Kutta method of order 4
- MATLAB's adaptive stepsize routines
- Systems of equations
- Higher order ODEs

Application: Newton's Law of Motion

Newton's Law of Motion is

$$F = ma$$

Acceleration is the time derivative of velocity, so

$$\frac{dv}{dt} = a$$

and

$$\frac{dv}{dt} = \frac{F}{m}$$

If $F(t)$ and $v(0)$ are known, we can (at least in principle) integrate the preceding equation to find $v(t)$

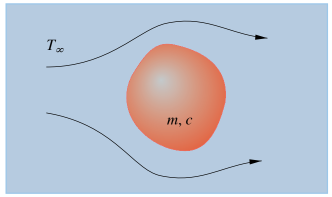
Application: Newton's Law of Cooling

The cooling rate of an object immersed in a flowing fluid is

$$Q = hA(T_s - T_\infty)$$

where Q is the heat transfer rate, h is the heat transfer coefficient, A is the surface area, T_s is the surface temperature, and T_∞ is the temperature of the fluid.

When the cooling rate is primarily controlled by the convection from the surface, the variation of the object's temperature with is described by an ODE.



Newton's Law of Cooling

Apply an energy balance

$$mc\frac{dT}{dt} = -Q = -hA(T_s - T_\infty)$$

Assume material is highly conductive $\Rightarrow T_s = T$

$$mc\frac{dT}{dt} = -hA(T - T_\infty)$$

or

$$\frac{dT}{dt} = -\frac{hA}{mc}(T - T_\infty)$$

Example: Analytical Solution

The ODE

$$\frac{dy}{dt} = -y \qquad y(0) = y_0$$

can be integrated directly:

$$\begin{aligned}
\frac{dy}{y} &= -dt \\
\ln y &= -t + C \\
\ln y - \ln C_2 &= -t \\
\ln \frac{y}{C_2} &= -t \\
y &= C_2 e^{-t} \\
y &= y_0 e^{-t}
\end{aligned}$$

Numerical Integration of First Order ODEs (1)

The generic form of a first order ODE is

$$\frac{dy}{dt} = f(t, y); \qquad y(0) = y_0$$

where the right hand side $f(t, y)$ is any single-valued function of t and y.

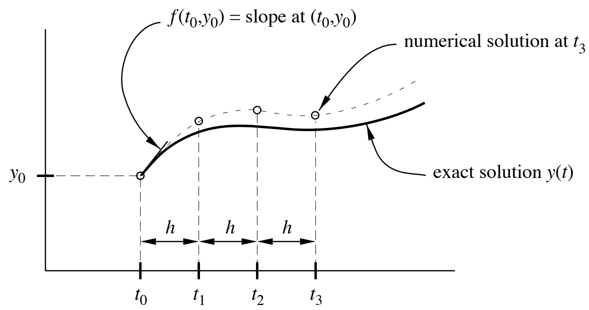
The approximate numerical solution is obtained at discrete values of t

$$t_j = t_0 + jh$$

where h is the "stepsize"

Numerical Integration of ODEs (2)

Graphical Interpretation



Nomenclature

$$
\begin{aligned}
y(t) &= \text{exact solution} \\
y(t_j) &= \text{exact solution evaluated at } t_j \\
y_j &= \text{approximate solution at } t_j \\
f(t_j, y_j) &= \text{approximate r.h.s. at } t_j
\end{aligned}
$$

Euler's Method (1)

Consider a Taylor series expansion in the neighborhood of t_0

$$y(t) = y(t_0) + (t - t_0)\left.\frac{dy}{dt}\right|_{t_0} + \frac{(t-t_0)^2}{2}\left.\frac{d^2y}{dt^2}\right|_{t_0} + \ldots$$

Retain only first derivative term and define

$$f(t_0, y_0) \equiv \left.\frac{dy}{dt}\right|_{t_0}$$

to get

$$y(t) \approx y(t_0) + (t - t_0) f(t_0, y_0)$$

Euler's Method (2)

Given $h = t_1 - t_0$ and initial condition, $y = y(t_0)$, compute

$$
\begin{aligned}
y_1 &= y_0 + h\, f(t_0, y_0) \\
y_2 &= y_1 + h\, f(t_1, y_1) \\
\vdots & \qquad \vdots \\
y_{j+1} &= y_j + h\, f(t_j, y_j)
\end{aligned}
$$

or

$$\boxed{y_j = y_{j-1} + h\, f(t_{j-1}, y_{j-1})}$$

Example: Euler's Method

Use Euler's method to integrate

$$\frac{dy}{dt} = t - 2y \qquad y(0) = 1$$

The exact solution is

$$y = \frac{1}{4}\left[2t - 1 + 5e^{-2t}\right]$$

j	t_j	$f(t_{j-1}, y_{j-1})$	Euler $y_j = y_{j-1} + h\,f(t_{j-1}, y_{j-1})$	Exact $y(t_j)$	Error $y_j - y(t_j)$
0	0.0	NA	(initial condition) 1.0000	1.0000	0
1	0.2	$0 - (2)(1) = -2.000$	$1.0 + (0.2)(-2.0) = 0.6000$	0.6879	-0.0879
2	0.4	$0.2 - (2)(0.6) = -1.000$	$0.6 + (0.2)(-1.0) = 0.4000$	0.5117	-0.1117
3	0.6	$0.4 - (2)(0.4) = -0.400$	$0.4 + (0.2)(-0.4) = 0.3200$	0.4265	-0.1065

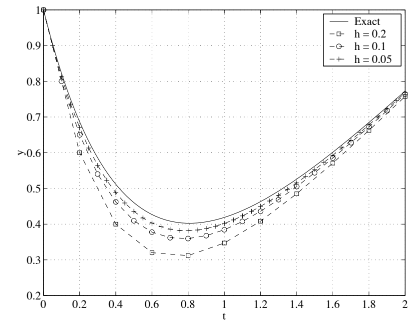
Reducing Stepsize Improves Accuracy (1)

Use Euler's method to integrate

$$\frac{dy}{dt} = t - 2y; \qquad y(0) = 1$$

for a sequence of smaller h (see `demoEuler`).

For a given h, the largest error in the numerical solution is the *Global Discretization Error* or *GDE*.



Reducing Stepsize Improves Accuracy (2)

Local error at any time step is

$$e_j = y_j - y(t_j)$$

where $y(t_j)$ is the exact solution evaluated at t_j.

$$\text{GDE} = \max(e_j), \qquad j = 1, \ldots$$

For Euler's method, GDE decreases linearly with h.

Here are results for the sample problem plotted on previous slide:

$$dy/dt = t - 2y; \quad y(0) = 1$$

h	$\max(e_j)$
0.200	0.1117
0.100	0.0502
0.050	0.0240
0.025	0.0117

Implementation of Euler's Method

```
function [t,y] = odeEuler(diffeq,tn,h,y0)
% odeEuler  Euler's method for integration of a single, first order ODE
%
% Synopsis:   [t,y] = odeEuler(diffeq,tn,h,y0)
%
% Input:      diffeq = (string) name of the m-file that evaluates the right
%                      hand side of the ODE written in standard form
%             tn  = stopping value of the independent variable
%             h   = stepsize for advancing the independent variable
%             y0  = initial condition for the dependent variable
%
% Output:     t = vector of independent variable values:  t(j) = (j-1)*h
%             y = vector of numerical solution values at the t(j)

t = (0:h:tn)';           %  Column vector of elements with spacing h
n = length(t);           %  Number of elements in the t vector
y = y0*ones(n,1);        %  Preallocate y for speed

%   Begin Euler scheme; j=1 for initial condition
for j=2:n
   y(j) = y(j-1) + h*feval(diffeq,t(j-1),y(j-1));
end
```

Analysis of Euler's Method (1)

Rewrite the discrete form of Euler's method as

$$\frac{y_j - y_{j-1}}{h} = f(t_{j-1}, y_{j-1}) \qquad \text{(discrete)}$$

Compare with original ODE

$$\frac{dy}{dt} = f(t, y) \qquad \text{(continuous)}$$

Substitute the exact solution into the discrete approximation to the ODE to get

$$\frac{y(t_j) - y(t_{j-1})}{h} - f(t_{j-1}, y(t_{j-1})) \neq 0$$

Analysis of Euler's Method (2)

Introduce a family of functions $z_j(t)$, which are the exact solutions to the ODE given the approximiate solution produced by Euler's method at step j.

$$\frac{dz_j}{dt} = f(t, y); \qquad z_j(t_{j-1}) = y_{j-1}$$

Due to truncation error, Euler's method produces a value of y_{j+1} that is different from $z_j(t_{j+1})$ even though by design, $y_j = z(t_j)$.

In other words

$$y_{j+1} - z(t_{j+1}) \neq 0$$

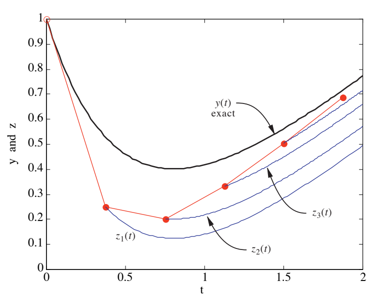
because y_{j+1} contains truncation error.

Analysis of Euler's Method (3)

Example: Solve

$$\frac{dy}{dt} = t - 2y; \qquad y(0) = 1$$

The plot shows the numerical solution (•) obtained with $h = 0.5$. The $z(t)$ curve starting at each of the numerical solution points is shown as a solid line.



Analysis of Euler's Method (4)

The **local discretization error** (LDE) is the residual obtained when the exact $z_j(t)$ is substituted into the discrete approximation to the ODE

$$\tau(t, h) = \frac{z(t_j) - z(t_{j-1})}{h} - f(t_{j-1}, z_j(t_{j-1}))$$

Note:

- User chooses h, and this affects LDE
- LDE also depends on t, the position in the interval

Analysis of Euler's Method (5)

Using a Taylor series expansion for $z_j(x)$ we find that

$$\frac{z(t_j) - z(t_{j-1})}{h} - f(t_{j-1}, z_j(t_{j-1})) = \frac{h}{2} z_j''(\xi)$$

where $t_{j-1} \leq \xi \leq t_j$ and $z_j'' \equiv d^2 z_j / dt^2$.

Thus, for Euler's method the local discretization error is

$$\tau(t, h) = \frac{h}{2} z_j''(\xi)$$

Since ξ is not known, the value of $z_j''(\xi)$ cannot be computed.

Analysis of Euler's Method (6)

Assume that $z_j''(\xi)$ is bounded by M in the interval $t_0 \leq t \leq t_N$. Then

$$\tau(t, h) \leq \frac{hM}{2} \quad \text{LDE for Euler's method}$$

Although M is unknown we can still compute the effect of reducing the stepsize by taking the ratio of $\tau(t, h)$ for two different choices of h

$$\boxed{\frac{\tau(t, h_2)}{\tau(t, h_1)} = \frac{h_2}{h_1}}$$

Global Discretization Error for Euler's Method

General application of Euler's method requires several steps to compute the solution to the ODE in an interval, $t_0 \leq t \leq t_N$. The local truncation error at each step accumulates. The result is the **global discretization error** (GDE)

The GDE for Euler's method is $\mathcal{O}(h)$. Thus

$$\boxed{\frac{\mathrm{GDE}(h_1)}{\mathrm{GDE}(h_2)} = \frac{h_1}{h_2}}$$

Summary of Euler's Method

Development of Euler's method has demonstrated the following *general* ideas

- The numerical integration scheme is derived from a truncated Taylor series approximation of the ODE.
- The local discretization error (LDE) accounts for the error at each time step.

  $$LDE = \mathcal{O}(h^p)$$

  where h is the stepsize and p is an integer $p \geq 1$.
- The global discretization error (GDE) includes the accumulated effect of the LDE when the ODE integration scheme is applied to an interval using several steps of size h.

  $$GDE = \mathcal{O}(h^p)$$

- The implementation separates the logic of the ODE integration scheme from the evaluation of the right hand side, $f(t, y)$. A general purpose ODE solver requires the user to supply a small m-file for evaluating $f(t, y)$.

Higher Order Methods

We now commence a survey of one-step methods that are more accurate than Euler's method.

- Not all methods are represented here
- Objective is a logical progression leading to RK-4
- Sequence is in order of increasing accuracy *and* increasing computational efficiency

Methods with increasing accuracy, lower GDE

Method	GDE
Euler	$\mathcal{O}(h)$
Midpoint	$\mathcal{O}(h^2)$
Heun	$\mathcal{O}(h^2)$
RK-4	$\mathcal{O}(h^4)$

Note that since $h < 1$, a GDE of $\mathcal{O}(h^4)$ is much smaller than a GDE of $\mathcal{O}(h)$.

Midpoint Method (1)

Increase accuracy by evaluating slope twice in each step of size h

$$k_1 = f(t_j, y_j)$$

Compute a tentative value of y at the midpoint

$$y_{j+1/2} = y_j + \frac{h}{2} f(t_j, y_j)$$

re-evaluate the slope

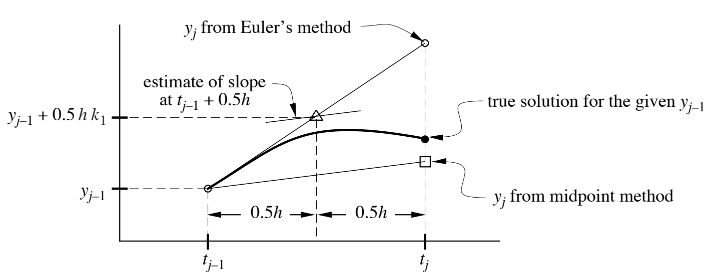
$$k_2 = f(t_j + \frac{h}{2}, y_j + \frac{h}{2} k_1)$$

Compute final value of y at the end of the full interval

$$y_{j+1} = y_j + h k_2$$

$\mathrm{LDE} = \mathrm{GDE} = \mathcal{O}(h^2)$

Midpoint Method (2)




```
function [t,y] = odeMidpt(diffeq,tn,h,y0)
% odeMidpt   Midpoint method for integration of a single, first order ODE
%
% Synopsis:  [t,y] = odeMidpt(diffeq,tn,h,y0)
%
% Input:     diffeq = (string) name of the m-file that evaluates the right
%                     hand side of the ODE written in standard form
%            tn  = stopping value of the independent variable
%            h   = stepsize for advancing the independent variable
%            y0  = initial condition for the dependent variable
%
% Output:    t = vector of independent variable values:  t(j) = (j-1)*h
%            y = vector of numerical solution values at the t(j)

t = (0:h:tn)';           %  Column vector of elements with spacing h
n = length(t);           %  Number of elements in the t vector
y = y0*ones(n,1);        %  Preallocate y for speed
h2 = h/2;                %  Avoid repeated evaluation of this constant

%   Begin Midpoint scheme; j=1 for initial condition
for j=2:n
   k1 = feval(diffeq,t(j-1),y(j-1));
   k2 = feval(diffeq,t(j-1)+h2,y(j-1)+h2*k1);
   y(j) = y(j-1) + h*k2;
end
```

Midpoint Method (3)

Midpoint method requires twice as much work per time step. Does the extra effort pay off?

Consider integration with Euler's method and $h = 0.1$. Formal accuracy is $\mathcal{O}(0.1)$.

Repeat calculations with the midpoint method and $h = 0.1$. Formal accuracy is $\mathcal{O}(0.01)$.

For Euler's method to obtain the same accuracy, the stepsize would have to be reduced by a factor of 10. The midpoint method, therefore, achieves the same (formal) accuracy with one fifth the work!

Comparison of Midpoint Method with Euler's Method

Solve

$$\frac{dy}{dt} = -y; \qquad y(0) = 1; \qquad 0 \le t \le 1$$

The exact solution is $y = e^{-t}$.

```
>> compEM

   h      nrhsE     errE      nrhsM     errM
 0.20000      6    4.02e-02      12    2.86e-03
 0.10000     11    1.92e-02      22    6.62e-04
 0.05000     21    9.39e-03      42    1.59e-04
 0.02500     41    4.65e-03      82    3.90e-05
 0.01250     81    2.31e-03     162    9.67e-06
 0.00625    161    1.15e-03     322    2.41e-06
```

For comparable accuracy:

- ➢ Midpoint method with $h = 0.2$ evaluates the right hand side of the ODE 12 times, and gives max error of 2.86×10^{-3}
- ➢ Euler's method with $h = 0.0125$ evaluates the right hand side of the ODE 81 times, and gives max error of 2.31×10^{-3}

Heun's Method (1)

Compute the slope at the starting point

$$k_1 = f(t_j, y_j)$$

Compute a tentative value of y at the endpoint

$$y_j^* = y_j + h f(t_j, y_j)$$

re-evaluate the slope

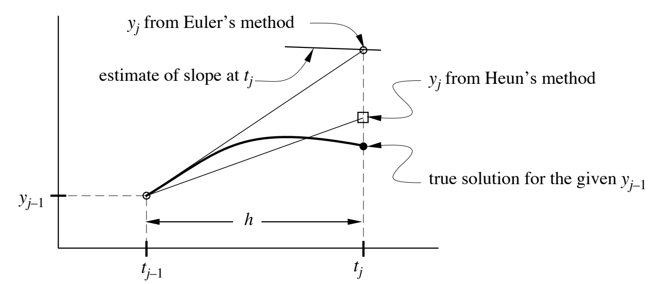
$$k_2 = f(t_j + h, y_j^*) = f(t_j + j, y_j + h k_1)$$

Compute final value of y with an average of the two slopes

$$y_{j+1} = y_j + h\,\frac{k_1 + k_2}{2}$$

$\mathrm{LDE} = \mathrm{GDE} = \mathcal{O}(h^2)$

Heun's Method (2)



Summary So Far

- Euler's method evaluates slope at beginning of the step
- Midpoint method evaluates slope at beginning and at midpoint of the step
- Heun's method evaluates slope at beginning and at end of step

Can we continue to get more accurate schemes by evaluating the slope at more points in the interval? Yes, but there is a limit beyond which additional evaluations of the slope increase in cost (increased flops) faster than the improve the accuracy.

Runge-Kutta Methods

Generalize the idea embodied in Heun's method. Use a *weighted average of the slope* evaluated at multiple in the step

$$y_{j+1} = y_j + h \sum \gamma_m k_m$$

where γ_m are weighting coefficients and k_m are slopes evaluated at points in the interval $t_j \le t \le t_{j+1}$

In general,

$$\sum \gamma_m = 1$$

Fourth Order Runge-Kutta

Compute slope at four places within each step

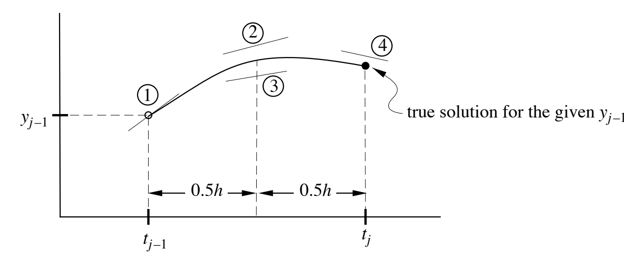
$$k_1 = f(t_j, y_j)$$
$$k_2 = f(t_j + \frac{h}{2}, y_j + \frac{h}{2}k_1)$$
$$k_3 = f(t_j + \frac{h}{2}, y_j + \frac{h}{2}k_2)$$
$$k_4 = f(t_j + h, y_j + hk_3)$$

Use weighted average of slopes to obtain y_{j+1}

$$y_{j+1} = y_j + h\left(\frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6}\right)$$

$\mathrm{LDE} = \mathrm{GDE} = \mathcal{O}(h^4)$

Fourth Order Runge-Kutta



```
function [t,y] = odeRK4(diffeq,tn,h,y0)
% odeRK4  Fourth order Runge-Kutta method for a single, first order ODE
%
% Synopsis:  [t,y] = odeRK4(fun,tn,h,y0)
%
% Input:     diffeq = (string) name of the m-file that evaluates the right
%                     hand side of the ODE written in standard form
%            tn  = stopping value of the independent variable
%            h   = stepsize for advancing the independent variable
%            y0  = initial condition for the dependent variable
%
% Output:    t = vector of independent variable values:  t(j) = (j-1)*h
%            y = vector of numerical solution values at the t(j)

t = (0:h:tn)';                  %  Column vector of elements with spacing h
n = length(t);                  %  Number of elements in the t vector
y = y0*ones(n,1);               %  Preallocate y for speed
h2 = h/2;  h3 = h/3;  h6 = h/6; %  Avoid repeated evaluation of constants

%  Begin RK4 integration; j=1 for initial condition
for j=2:n
   k1 = feval(diffeq, t(j-1),    y(j-1)        );
   k2 = feval(diffeq, t(j-1)+h2, y(j-1)+h2*k1  );
   k3 = feval(diffeq, t(j-1)+h2, y(j-1)+h2*k2  );
   k4 = feval(diffeq, t(j-1)+h,  y(j-1)+h*k3   );
   y(j) = y(j-1) + h6*(k1+k4) + h3*(k2+k3);
end
```


Comparison of Euler, Midpoint and RK4 (1)

Solve

$$\frac{dy}{dt} = -y; \qquad y(0) = 1; \qquad 0 \le t \le 1$$

```
>> compEMRK4

   h       nrhsE     errE      nrhsM     errM      nrhsRK4    err4
 0.20000       6   4.02e-02      12    2.86e-03       24    5.80e-06
 0.10000      11   1.92e-02      22    6.62e-04       44    3.33e-07
 0.05000      21   9.39e-03      42    1.59e-04       84    2.00e-08
 0.02500      41   4.65e-03      82    3.90e-05      164    1.22e-09
 0.01250      81   2.31e-03     162    9.67e-06      324    7.56e-11
 0.00625     161   1.15e-03     322    2.41e-06      644    4.70e-12
```


Comparison of Euler, Midpoint and RK4 (2)

	Error	step size	RHS evaluations
Euler	1.2×10^{-3}	0.00625	161
Midpoint	2.9×10^{-3}	0.2	12
Midpoint	2.4×10^{-6}	0.00625	322
RK-4	5.8×10^{-6}	0.2	24

Conclusion:

➢ RK-4 is much more accurate (smaller GDE) than Midpoint or Euler

➢ Although RK-4 takes more flops per step, it can achieve comparable accuracy with much larger time steps. The net effect is that **RK-4 is more accurate and more efficient**

Summary: Accuracy of ODE Integration Schemes

- GDE decreases as h decreases
- Need an upper limit on h to achieve a desired accuracy
- Example: Euler's Method

$$y_j = y_{j-1} + h\, f(t_{j-1}, y_{j-1})$$

  when h and $|f(t_j, y_j)|$ are large, the change in y is large
- The product, $h\, f(t_j, y_j)$, determines accuracy

Procedure for Using Algorithms Having Fixed Stepsize

- Develop an m-file to evaluate the right hand side
- Use a high order method, e.g. RK-4
- Compare solutions for a sequence of smaller h
- When the change in the solution between successively smaller h is "small enough", accept that as the h-*independent* solution.

The goal is to obtain a solution that does not depend (in a significant way) on h.

Adaptive Stepsize Algorithms

Let the solution algorithm determine h *at each time step*

- Set a tolerance on the error
- When $|f(t_{j-1}, y_{j-1})|$ is decreases, increase h to increase efficiency and decrease round-off
- When $|f(t_{j-1}, y_{j-1})|$ is increases, decrease h to maintain accuracy

Adaptive Stepsize Algorithms (2)

How do we find the "error" at each step in order to judge whether the stepsize needs to be reduced or increased?

Two related strategies:

➢ Use two h values at each step:
  1. Advance the solution with $h = h_1$
  2. Advance the solution with two steps of size $h_2 = h/2$
  3. If solutions are close enough, accept the h_1 solution, stop
  4. Otherwise, replace $h_1 = h_2$, go back to step 2

➢ Use *embedded* Runge-Kutta methods

Embedded Runge-Kutta Methods (1)

There is a **pair of RK methods** that use the **same six k values**

Fourth Order RK:

$$
\begin{aligned}
y_{j+1} = y_j &+ c_1k_1 + c_2k_2 + c_3k_3 \\
&+ c_4k_4 + c_5k_5 + c_6k_6 + \mathcal{O}(h^4)
\end{aligned}
$$

Fifth Order RK:

$$
\begin{aligned}
y^*_{j+1} = y_j &+ c^*_1k_1 + c^*_2k_2 + c^*_3k_3 \\
&+ c^*_4k_4 + c^*_5k_5 + c^*_6k_6 + \mathcal{O}(h^5)
\end{aligned}
$$

Therefore, at each step an estimate of the truncation error is

$$\Delta = y_{j+1} - y^*_{j+1}$$

Embedded Runge-Kutta Methods (2)

Possible outcomes

- If Δ is smaller than tolerance, accept the y_{j+1} solution.
- If Δ is *much* smaller than tolerance, accept the y_{j+1} solution, *and* try increasing the stepsize.
- If Δ is larger than tolerance, reduce h and try again.

MATLAB ode45 Function

- User supplies error tolerance, *not* stepsize
- Simultaneously compute 4^{th} and 5^{th} order Runge-Kutta solution
- Compare two solutions to determine accuracy
- Adjust step-size so that error tolerance is maintained

MATLAB ode45 Function (2)

Error tolerances are

$$\tilde{\tau} < \max(\texttt{RelTol} \times |y_j|, \texttt{AbsTol})$$

where $\tilde{\tau}$ is an estimate of the local truncation error, and RelTol and AbsTol are the error tolerances, which have the default values of

$$\texttt{RelTol} = 1 \times 10^{-3} \qquad \texttt{AbsTol} = 1 \times 10^{-6}$$

Using ode45

Syntax:

```
[t,Y] = ode45(diffeq,tn,y0)
[t,Y] = ode45(diffeq,[t0 tn],y0)
[t,Y] = ode45(diffeq,[t0 tn],y0,options)
[t,Y] = ode45(diffeq,[t0 tn],y0,options,arg1,arg2,...)
```

User writes the *diffeq* m-file to evaluate the right hand side of the ODE.

Solution is controlled with the odeset function

```
options = odeset('parameterName',value,...)
[y,t] = ode45('rhsFun',[t0 tN],y0,options,...)
```

Syntax for ode23 and other solvers is the same.

MATLAB's Built-in ODE Routines

Function	Description
`ode113`	Variable order solution to nonstiff systems of ODEs. `ode113` uses an explicit predictor-corrector method with variable order from 1 to 13.
`ode15s`	Variable order, multistep method for solution to stiff systems of ODEs. `ode15s` uses an implicit multistep method with variable order from 1 to 5.
`ode23`	Lower order adaptive stepsize routine for non-stiff systems of ODEs. `ode23` uses Runge-Kutta schemes of order 2 and 3.
`ode23s`	Lower order adaptive stepsize routine for moderately stiff systems of ODEs. `ode23` uses Runge-Kutta schemes of order 2 and 3.
`ode45`	Higher order adaptive stepsize routine for non-stiff systems of ODEs. `ode45` uses Runge-Kutta schemes of order 4 and 5.

Interpolation Refinement by `ode45` (1)

The `ode45` function attempts to obtain the solution to within the user-specified error tolerances. In some situations the solution can be obtained within the tolerance by taking so few time steps that the solution appears to be unsmooth. To compensate for this, `ode45` automatically interpolates the solution between points that are obtained from the solver.

Consider

$$\frac{dy}{dt} = \cos(t), \qquad y(0) = 0$$

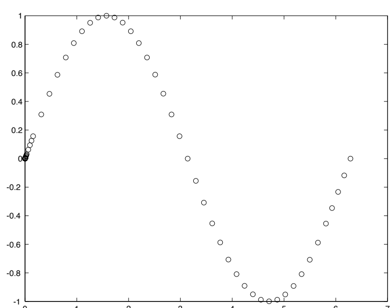
The following statements obtain the solution with the default parameters.

```
>> rhs = inline('cos(t)','t','y');
>> [t,Y] = ode45(rhs,[0 2*pi],0);
>> plot(t,Y,'o')
```

(See plot on next slide)

Interpolation Refinement by `ode45` (2)

```
>> rhs = inline('cos(t)','t','y');
>> [t,Y] = ode45(rhs,[0 2*pi],0);
>> plot(t,Y,'o')
```



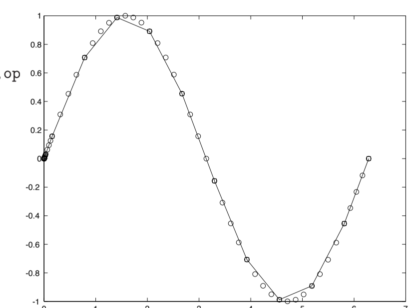
Interpolation Refinement by `ode45` (3)

Repeat without interpolation:

```
>> options = odeset('Refine',1)
>> [t2,Y2] = ode45(rhs,[0 2*pi],0,op
>> hold on
>> plot(t2,Y2,'rs-')
```

The two solutions are identical at the points obtained from the Runge-Kutta algorithm

```
>> max(Y2 - Y(1:4:end))
ans =
     0
```



Coupled ODEs (1)

Consider

$$\frac{dy_1}{dt} = f_1(t, y_1, y_2)$$
$$\frac{dy_2}{dt} = f_2(t, y_1, y_2)$$

These equations must be advanced simultaneously.

Coupled ODEs (2)

Apply the 4^{th} Order Runge-Kutta Scheme:

$$k_{1,1} = f_1\left(t_j, y_{j,1}, y_{j,2}\right)$$
$$k_{1,2} = f_2\left(t_j, y_{j,1}, y_{j,2}\right)$$

$$k_{2,1} = f_1\left(t_j + \frac{h}{2}, y_{j,1} + \frac{h}{2}k_{1,1}, y_{j,2} + \frac{h}{2}k_{1,2}\right)$$
$$k_{2,2} = f_2\left(t_j + \frac{h}{2}, y_{j,1} + \frac{h}{2}k_{1,1}, y_{j,2} + \frac{h}{2}k_{1,2}\right)$$

$$k_{3,1} = f_1\left(t_j + \frac{h}{2}, y_{j,1} + \frac{h}{2}k_{2,1}, y_{j,2} + \frac{h}{2}k_{2,2}\right)$$
$$k_{3,2} = f_2\left(t_j + \frac{h}{2}, y_{j,1} + \frac{h}{2}k_{2,1}, y_{j,2} + \frac{h}{2}k_{2,2}\right)$$

$$k_{4,1} = f_1\left(t_j + h, y_{j,1} + hk_{3,1}, y_{j,2} + hk_{3,2}\right)$$
$$k_{4,2} = f_2\left(t_j + h, y_{j,1} + hk_{3,1}, y_{j,2} + hk_{3,2}\right)$$

Coupled ODEs (3)

Update y_1 and y_2 only after all slopes are computed

$$y_{j+1,1} = y_{j,1} + h\left(\frac{k_{1,1}}{6} + \frac{k_{2,1}}{3} + \frac{k_{3,1}}{3} + \frac{k_{4,1}}{6}\right)$$
$$y_{j+1,2} = y_{j,2} + h\left(\frac{k_{1,2}}{6} + \frac{k_{2,2}}{3} + \frac{k_{3,2}}{3} + \frac{k_{4,2}}{6}\right)$$

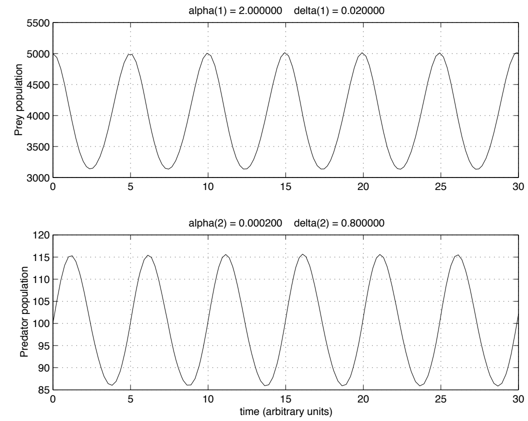
Example: Predator-Prey Equations

$$\frac{dp_1}{dt} = \alpha_1 p_1 - \delta_1 p_1 p_2 \qquad \text{(prey)}$$
$$\frac{dp_2}{dt} = \alpha_2 p_1 p_2 - \delta_2 p_2 \qquad \text{(preditor)}$$

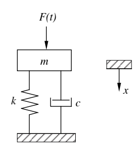
Evaluate RHS of Preditor-Prey Model

```
function dpdt = rhsPop2(t,p,flag,alpha,delta)
% rhsPop2  Right hand sides of coupled ODEs for 2 species predator-prey system
%
% Synopis:  dpdt = rhsPop2(t,p,flag,alpha,delta)
%
% Input:    t = time. Not used in this m-file, but needed by ode45
%           p = vector (length 2) of populations of species 1 and 2
%           flag  = (not used) placeholder for compatibility with ode45
%           alpha = vector (length 2) of growth coefficients
%           delta = vector (length 2) of mortality coefficients
%
% Output:   dpdt = vector of dp/dt values
dpdt = [ alpha(1)*p(1)      - delta(1)*p(1)*p(2);
         alpha(2)*p(1)*p(2) - delta(2)*p(2);      ];
```


Preditor-Prey Results



Example: Second Order Mechanical System



$$\sum F = ma$$

Forces acting on the mass are

$$F_{spring} = -kx$$

$$F_{damper} = -c\dot{x}$$

$$F(t) - kx - c\dot{x} = m\ddot{x}$$

Second Order Mechanical System

Governing equation is a second order ODE

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F}{m}$$

$$\zeta \equiv \frac{c}{2\sqrt{km}}$$

$$\omega_n \equiv \sqrt{k/m}$$

ζ and ω_n are the only (dimensionless) parameters

Equivalent Coupled First Order Systems

Define

$$y_1 \equiv x \qquad y_2 \equiv \dot{x}$$

then

$$\begin{aligned}\frac{dy_1}{dt} &= \dot{x} = y_2 \\ \frac{dy_2}{dt} &= \ddot{x} \\ &= \frac{F}{m} - 2\zeta\omega_n\dot{x} - \omega_n^2 x \\ &= \frac{F}{m} - 2\zeta\omega_n y_2 - \omega_n^2 y_1\end{aligned}$$

Solve Second Order System with ODE45

```
function demoSmd(zeta,omegan,tstop)
% demoSmd   Second order system of ODEs for a spring-mass-damper system
%
% Synopsis:  smdsys(zeta,omegan,tstop)
%
% Input:     zeta   = (optional) damping ratio;  Default:  zeta = 0.1
%            omegan = (optional) natural frequency;  Default: omegan = 35
%            tstop  = (optional) stopping time;  Default:  tstop = 1.5
%
% Output:    plot of displacement and velocity versus time

if nargin<1,  zeta = 0.1;   end
if nargin<2,  omegan = 35;  end
if nargin<3,  tstop = 1.5;  end

y0 = [0; 0];   a0 = 9.8;     %  Initial conditions and one g force/mass
[t,y] = ode45('rhssmd',tstop,y0,[],zeta,omegan,a0);

subplot(2,1,1);
plot(t,y(:,1));  ylabel('Displacement');  grid;
title(sprintf('zeta = %5.3f   omegan = %5.1f',zeta,omegan));
subplot(2,1,2);
plot(t,y(:,2));  xlabel('Time (s)');  ylabel('Velocity');  grid;
```

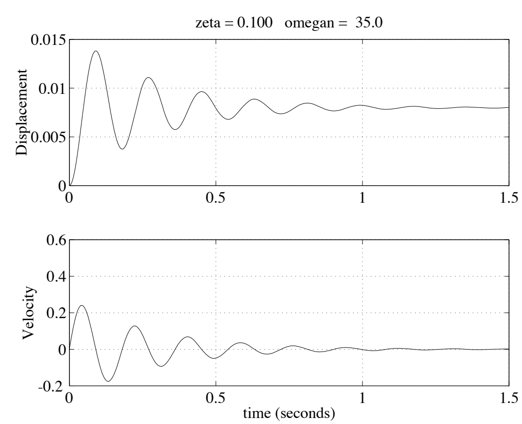

Solve Second Order System with ODE45

```
function dydt = rhsSmd(t,y,flag,zeta,omegan,a0)
% rhsSmd  Right-hand sides of coupled ODEs for a spring-mass-damper system
%
% Synopis:  dydt = rhsSmd(t,y,flag,zeta,omegan,a0)
%
% Input:    t      = time, the independent variable
%           y      = vector (length 2) of dependent variables
%                    y(1) = displacement and y(2) = velocity
%           flag   = dummy argument for compatibility with ode45
%           zeta   = damping ratio (dimensionless)
%           omegan = natural frequency (rad/s)
%           a0     = input force per unit mass
%
% Output:   dydt = column vector of dy(i)/dt values

if t<=0, fonm = 0.0;
else,    fonm = a0;    %  Force/mass (acceleration)
end

dydt = [ y(2);  fonm - 2*zeta*omegan*y(2) - omegan*omegan*y(1)];
```


Response of Second Order System to a Step Input



General Procedure for Higher Order ODEs

Given

$$\frac{d^n u}{dt^n} = f(t, u)$$

The transformation is

Define y_i	ODE for y_i
$y_1 = u$	$\frac{dy_1}{dt} = y_2$
$y_2 = \frac{du}{dt}$	$\frac{dy_2}{dt} = y_3$
$y_3 = \frac{d^2u}{dt^2}$	$\frac{dy_3}{dt} = y_4$
$\vdots$	$\vdots$
$y_n = \frac{d^{n-1}u}{dt^{n-1}}$	$\frac{dy_n}{dt} = f(t, u)$